Matrix Perturbation for Structural Dynamic Analysis

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The objective of the effort was to investigate methodologies to reduce the cost of evaluating changes in dynamic loads when small modifications are made in the structure. A matrix perturbation technique has been developed to calculate the dynamic responses of a structural system that has been modified from the original design. The calculation is based on the results of the original design and the assumption that the structural modification is small. The advantage of the method is an update of the dynamic response caused by design changes without performing an entire analysis. This procedure can be used in a design load analysis cycle in which the structural design is subject to frequent changes. A sample problem is given to demonstrate the validity of the technique.

Introduction

FOR any structural system whose dynamic responses are an important consideration in its design, the responses of the representative analytical model caused by the relevant dynamic environments are required. In the design of structural systems, an iterative design/analysis process is performed until a satisfactory design is achieved. The cycle of the iterative process consists of an update of the response or load estimation, the structural design activity, the math model preparation or revision, and, again a response analysis. The process will be referred to as a design load analysis cycle (DLAC). The number of DLACs more often is influenced by the number of design changes than by different load considerations.

A DLAC is expensive because the mathematical models are large and the time delay between the availability of the model and the final dynamic response results is substantial. The cost of a DLAC is increased substantially in the more common multiorganization effort necessary to generate responses. In most aerospace programs, the integrated payload model is sent to a launch vehicle integration organization to perform the response analysis resulting from the various dynamic environments. Occasionally, as many as ten organizations are involved and a DLAC may require up to six months.[1]

A substantial cost savings can be realized if a quick evaluation of a change in payload responses resulting from payload design changes is possible. The information provides timely payload data for the tradeoff and design impact studies as well as a guide in the decision for another complete DLAC. The payload designer can make these evaluations independent of the launch vehicle because, in most cases, the design of the payload is changed whereas that of the launch vehicle is not.

The present investigation will show that the results of the modified model can be obtained without repeating the entire DLAC, if the design changes are small. A small design change does not necessarily imply a small change in its response, and the change in response is related to the characteristics of the dynamic environment.

Approach

One powerful method for solving nonlinear differential equations is the perturbation method. The use of this method initially was limited to astronomical calculations, but important contributions of Poincaré[2] and later mathematicians have extended the perturbation method for the solution of a more general class of problems in the field of mechanics.

Recently, Caughey has applied the matrix perturbation technique to the design of subsystems in large complex structural[3] and other dynamical problems.[4] Chen and Wada obtained analysis-test correlation criteria for structural dynamic systems by using the matrix perturbation technique.[5] In application, it consists of developing the desired quantities in powers of a small parameter multiplied by coefficients that are functions of the independent variable. The perturbation quantities are determined individually, usually by solving a sequence of linear equations.

The governing equations of a structural dynamic system for the finite-element formulation may be expressed as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f(t)\} \tag{1}$$

where $[M]$, $[C]$, and $[K]$ are the mass, damping, and stiffness matrices, respectively, and $\{x\}$ and $\{f(t)\}$ are the displacement vector and external forcing function, respectively. The design changes in a structural system may be reflected by the changes in the mass, damping, and stiffness matrices. Since these changes are usually small compared to the entire system, the updated mass and stiffness matrices can be expressed as

$$[M]=[M_0]+\epsilon[M_1], \quad [K]=[K_0]+\epsilon[K_1] \tag{2}$$

where $[M_0]$ and $[K_0]$ are the original mass and stiffness matrices, respectively, and $\epsilon[M_1]$ and $\epsilon[K_1]$ are the corresponding changes. Here ϵ is a small parameter representing the deviations between the updated and the original model. As these deviations approach zero, $\epsilon \to 0$, the updated values approach the corresponding original values, $[M]\to[M_0]$, $[K]\to[K_0]$.

For simplicity, only the design changes of mass and stiffness matrices are considered in the present study. The changes in damping matrix can be incorporated similarly.

From Eq. (2), it is clear that the mass and stiffness matrices are analytical functions of the small parameter ϵ. According to the perturbation theory, the solutions of the governing Eq. (1) also can be expressed as the analytical function of ϵ,[6]

$$\{x\}=\{x_0\}+\epsilon\{x_1\}+\epsilon^2\{x_2\}+\ldots \tag{3}$$

where $\{x_0\}$ is the displacement response to the original model, and

$$[\phi]=[\phi_0]+\epsilon[\phi_1]+\epsilon^2[\omega_2]+\ldots \tag{4}$$

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$$[\lceil\omega\rfloor] = [\lceil\omega_0\rfloor] + \epsilon[\lceil\omega_1\rfloor] + \epsilon^2[\lceil\omega_2\rfloor] + \ldots \tag{5}$$

where $[\phi]$ and $[\omega]$ are eigenvectors and eigenvalues, respectively, of the modified model and $[\phi_0]$ and $\lceil\omega_0\rfloor$ are the corresponding values of the original model.

The objective of the present study is to calculate this updated solution, $\{x\}$, $[\phi]$, and $[\lceil\omega\rfloor]$, without re-solving the governing Eq. (1). Since the solutions of the original model, $\{x_0\}$, $[\phi_0]$, and $[\lceil\omega_0\rfloor]$ are known, only the perturbed values, $\{x_1\}$, $[\phi_1]$, $\lceil\omega_1\rfloor$, etc., are required.

From the orthogonality conditions of the eigenvectors and by neglecting the terms of $O(\epsilon^2)$, the following conditions may be obtained[5]:

$$[\alpha] + [\alpha]^T = -[\phi_0]^T[M_1][\phi_0] \tag{6}$$

$$2\lceil\omega_0\rfloor\lceil\omega_1\rfloor = \lceil\omega_0^2\rfloor[\alpha] + [\alpha]^T\lceil\omega_0^2\rfloor + [\phi_0]^T[K_1][\phi_0] \tag{7}$$

where

$$[\phi_1] = [\phi_0][\alpha] \tag{8}$$

With given modification of the mass and stiffness matrices, $[M_1]$ and $[K_1]$, the matrices $[\alpha]$ and $\lceil\omega_1\rfloor$ can be calculated from Eqs. (6) and (7). This in turn gives the first-order perturbation update of the eigenvectors and eigenvalues. Similar procedures can be applied to calculate the higher-order perturbation.

After the eigenvalues and eigenvectors of the modified model are obtained, the model's dynamic responses frequently are desired. The method of generalized coordinates will be used in this effort.

Let the displacement vector of the system $\{x\}$ be expressed as

$$\{x\} = [\phi]\{u\} \tag{9}$$

the governing equation for generalized coordinates $\{u\}$ may be written as

$$\{\ddot{u}\} + 2\lceil\rho\rfloor\lceil\omega\rfloor\{\dot{u}\} + \lceil\omega^2\rfloor\{u\} = \{F\} \tag{10}$$

where

$$2\lceil\rho\rfloor\lceil\omega\rfloor = [\phi]^T[C][\phi] \tag{11}$$

$$\{F\} = [\phi]^T\{f\}, \text{ generalized forcing function} \tag{12}$$

The system is assumed to possess proportional damping. Equation (10) is completely decoupled.

Now, since the coefficients and inhomogeneous part of Eq. (10) can be expressed as analytical functions of the small parameter ϵ, the dependent variable $\{u\}$ also can be expressed as an analytical function of ϵ. That is

$$\{u\} = \{u_0\} + \epsilon\{u_1\} + \epsilon^2\{u_2\} + \ldots \tag{13}$$

Upon substitution of Eqs. (3-5, and 13) into Eqs. (9) and (10), and equating the coefficients of like power of ϵ, one obtains

$$\epsilon^0: \left.\begin{aligned} &\{x_0\} = [\phi_0]\{u_0\} \\ &\{\ddot{u}_0\} + 2\lceil\rho\rfloor\lceil\omega_0\rfloor\{\dot{u}_0\} + \lceil\omega_0^2\rfloor\{u_0\} \\ &= [\phi_0]^T\{f\} = \{F_0\} \end{aligned}\right\} \tag{14}$$

$$\epsilon': \left.\begin{aligned} &\{x_1\} = [\phi_0\{u_1\} + [\phi_1]\{u_0\} \\ &\{\ddot{u}_1\} + 2\lceil\rho\rfloor\lceil\omega_0\rfloor\{\dot{u}_1\} + \lceil\omega_0^2\rfloor\{u_1\} \\ &= [\phi_1]^T\{f\} - 2\lceil\rho\rfloor\lceil\omega_1\rfloor\{\dot{u}_0\} \\ &-2\lceil\omega_0\rfloor[\omega_1\rfloor\{u_0\} \end{aligned}\right\} \tag{15}$$

$\epsilon^2: \ldots$

The percentage critical damping $\lceil\rho\rfloor$ is not subject to design change in the development, although it can be incorporated in a similar manner.

Equation (14) is identical to the original model whose solution is already known. Equation (15) can be solved once the solution of Eq. (14) is obtained.

Since the results are dependent on the type of forcing function, various forcing functions are considered. Note again that the objective is not to re-solve Eq. (15), but to obtain the solution of Eq. (15) using the solution of Eq. (14).

Monotonic Forcing Function

If the nature of a forcing function is monotonically increasing or decreasing, it always can be expressed as a summation of exponential functions. Let the forcing function be described as

$$\{F_0\} = [\phi_0]^T\{f(t)\} = \{ae^{\gamma t}\} \tag{16}$$

Equation (14) becomes

$$\ddot{u}_0 + 2\rho\omega_0\dot{u}_0 + \omega_0^2 u = ae^{\gamma t} \tag{17}$$

Since the governing equations for the generalized coordinates are decoupled, subsequently they will be written in the scalar form. The solution of Eq. (17) can be written as

$$u_0 = \frac{ae^{\gamma t}}{\gamma^2 + 2\rho\omega_0\gamma + \omega_0^2} \tag{18}$$

The first-order perturbation Eq. (15) can be written as

$$\ddot{u}_1 + 2\rho\omega_0\dot{u}_1 + \omega_0^2 u_1 = \left[\alpha^T - 2\omega_1\left(\frac{\rho\gamma + \omega_0}{\gamma^2 + 2\rho\omega_0\gamma + \omega_0^2}\right)\right]ae^{\gamma t} \tag{19}$$

and the solution as

$$u_1 = \left[\alpha^T - 2\omega_1\left(\frac{\rho\gamma + \omega_0}{\gamma^2 + 2\rho\omega_0\gamma + \omega_0^2}\right)\right]u_0 \tag{20}$$

Equation (20) indicates that the first-order perturbation solution, u_1 can be obtained by multiplying the original solution u_0, by a factor. This factor is a function of α and ω_1 that can be calculated once the modifications of mass and stiffness matrices are known as shown in Eqs. (6) and (7).

Periodic Forcing Function

Let

$$F_0 = a\sin\Omega t \tag{21}$$

Then the solution to Eq. (14) can be written as

$$u_0 = A_0\sin\Omega t + B_0\cos\Omega t \tag{22}$$

$$A_0 = \frac{\omega_0^2 - \Omega^2}{(\omega_0^2 - \Omega^2)^2 + 4\rho^2\omega_0^2\Omega^2}$$

$$B_0 = \frac{-2\rho\omega_0\Omega}{(\omega_0^2 - \Omega^2)^2 + 4\rho^2\omega_0^2\Omega^2} \tag{23}$$

Now the order of magnitude of the response will be examined at the resonant frequency.

At $\Omega = \omega_0$, from Eqs. (22) and (23),

$$u_0 = O(1/\rho\omega_0^2)$$

and from Eq. (15),

$$u_1 = O[1/(\rho\omega_0^2)^2]$$

For a system with very low damping and low natural frequency, the quantity $\rho\omega_0^2$ can be very small such that the condition $\epsilon u_1 << u_0$ will be violated. This indicates that the power series expansion of the small parameter ϵ is not uniformly valid within the domain of interest.

Lighthill[7] first recognized this problem and introduced a very important extension of Poincaŕe's original perturbation method. Kuo[8] further developed the method for solving the fluid mechanics boundary-layer problems. The principle is to expand not only the dependent variable but also the independent variables in a power series of the small parameter ϵ. An elegant paper by Tsien[9] reviewed this important technique showing the generality of the concept for calculating the approximate solution of a physical problem. This technique will be used in the present study.

For this purpose, the governing Eq. (10) will be rewritten as

$$\frac{dv}{dt}+2\rho\omega v+\omega^2 u=F, \quad \frac{du}{dt}=v \tag{24}$$

Then let

$$u(t)=u_0(\xi)+\epsilon u_1(\xi)+\epsilon^2 u_2(\xi)+\ldots \tag{25a}$$

$$v(t)=v_0(\xi)+\epsilon v_1(\xi)+\epsilon^2 v_2(\xi)+\ldots \tag{25b}$$

$$t=\xi+\epsilon\tau_1(\xi)+\epsilon^2\tau_2(\xi)+\ldots \tag{25c}$$

Upon substitution of Eqs. (5) and (25) into Eq. (24), one obtains

$$\begin{aligned}&\frac{dv_0}{d\xi}+\epsilon\frac{dv_1}{d\xi}+2\rho(\omega_0+\epsilon\omega_1)(v_0+\epsilon v_1)\frac{dt}{d\xi}+(\omega_0^2+2\epsilon\omega_0\omega_1)\\&\times(u_0+\epsilon u_1)\frac{dt}{d\xi}+O(\epsilon^2)=F(\xi+\epsilon\tau_1)\frac{dt}{d\xi}\\&=\left[F(\xi)+\epsilon\tau_1\frac{dF}{d\xi}\right]\frac{dt}{dt}+O(\epsilon^2)\end{aligned} \tag{26}$$

$$\left(\frac{du_0}{d\xi}+\epsilon\frac{du_1}{d\xi}\right)=(v_0+\epsilon v_1)\frac{dt}{d\xi}+O(\epsilon^2) \tag{27}$$

Also, by differentiating Eq. (25c), one obtains

$$\frac{dt}{d\xi}=1+\epsilon\frac{d\tau_1}{d\xi}+O(\epsilon^2) \tag{28}$$

Substituting Eq. (28) into Eqs. (26) and (27) and equating the coefficients of like powers of ϵ, one obtains

$$\epsilon^\circ: \left.\begin{aligned}&\frac{dv_0}{d\xi}+2\rho\omega_0 v_0+\omega_0^2 u_0=F_0(\xi)\\&\frac{du_0}{d\xi}=v_0\end{aligned}\right\} \tag{29}$$

$$\epsilon': \left.\begin{aligned}&\frac{dv_1}{d\xi}+2\rho\omega_0 v_1+\omega_0^2 u_1=\tau_1\frac{dF}{d\xi}+\frac{dv_0}{d\xi}\frac{d\tau_1}{d\xi}\\&-2\omega_1(\rho v_0+\omega_0 u_0)\\&\frac{du_1}{d\xi}=v_1+\frac{du_0}{d\xi}\frac{d\tau_1}{d\xi}\end{aligned}\right\} \tag{30}$$

The two equations in Eq. (29) can be combined as

$$\frac{d^2u_0}{d\xi^2}+2\rho\omega_0\frac{du_0}{d\xi}+\omega_0^2 u_0=F_0(\xi) \tag{31}$$

whose solution is the original solution expressed in the ξ domain instead of the t domain.

The two Eqs. (30) can be combined as

$$\begin{aligned}&\frac{d^2u_1}{d\xi^2}+2\rho\omega_0\frac{du_1}{d\xi}+\omega_0^2 u_1=2\frac{d\tau_1}{d\xi^2}\frac{d^2u_0}{d\xi^2}\\&+\left(2\rho\omega_0\frac{d\tau_1}{d\xi}+\frac{d^2\tau_1}{d\xi^2}\right)\frac{du_0}{d\xi}-2\omega_1\left(\rho\frac{du_0}{d\xi}+\omega_0 u_0\right)+\tau_1\frac{dF_0}{d\xi}\end{aligned} \tag{32}$$

Equation (32) has two unknowns, namely u_1 and τ_1. We would like to have τ_1 valued such that u_1 can be expressed as a function of u_0. For the case of a periodic forcing function, let τ_1 = constant, such that $d\tau_1/d\xi=d^2\tau_1/d\xi^2=0$.

Thus

$$\frac{d^2u_1}{d\xi^2}+2\rho\omega_0\frac{du_1}{d\xi}+\omega_0^2 u_1=\tau_1\frac{dF_0}{d\xi}-2\omega_1\left(\rho\frac{du_0}{d\xi}+\omega_0 u_0\right) \tag{33}$$

Let

$$u_1(\xi)=\beta u_0(\xi) \tag{34}$$

Upon substitution, one obtains

$$-\beta F_0+\tau_1\frac{dF_0}{d\xi}-2\omega_1\left(\rho\frac{du_0}{d\xi}+\omega_0 u_0\right)=0 \tag{35}$$

Using Eqs. (21) and (22), Eq. (35) can be written as

$$\begin{aligned}&[\Omega\tau_1-2\omega_1(\rho\Omega A_0+\omega_0 B_0)]\cos\Omega\xi+[-2\omega_1(\omega_0 A_0\\&-\rho\Omega B_0)-\beta]\sin\Omega\xi=0\end{aligned}$$

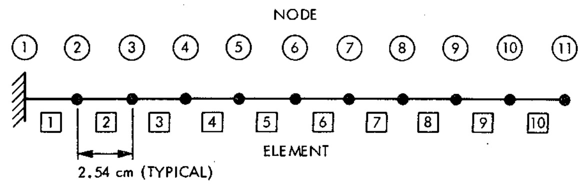
For this equation to be valid for all values of ϵ, the coefficients of $\cos\Omega\xi$ and $\sin\Omega\xi$ must vanish. Therefore,

$$\tau_1=-\frac{2\rho\omega_1(\omega_0^2+\Omega^2)}{(\omega_0^2+\Omega^2)^2+4\rho^2\omega_0^2\Omega^2} \tag{36}$$

$$\beta=\frac{2\omega_0\omega_1[(1-2\rho^2)\Omega^2-\omega_0^2]}{(\omega_0^2+\Omega^2)^2+4\rho^2\omega_0^2\Omega^2} \tag{37}$$

The updated solution can be obtained as follows:

$$\begin{aligned}&u(t)=u_0(t)+\epsilon u_1(t)+\ldots\cong(A_0\sin\Omega t+B_0\cos\Omega t)\\&+\epsilon[A_0\sin\Omega(t-\epsilon\tau_1)+B_0\cos\Omega(t-\epsilon\tau_1)]\end{aligned} \tag{38}$$



	MASS (kg - sec^2/cm)			MOMENT OF INERTIA (cm^4)	
NODE	MODEL I	MODEL II	ELEMENT	MODEL I	MODEL II
2	1.7858	1.9644	1	2.0812	2.2893
3	1.7858	1.9644	2	2.0812	2.2893
4	1.7858	2.1430	3	2.0812	2.2893
5	1.7858	2.1430	4	2.0812	1.8731
6	1.7858	2.3216	5	2.0812	1.8731
7	1.7858	2.1430	6	2.0812	1.8731
8	1.7858	2.1430	7	2.0812	1.6650
9	1.7858	2.1430	8	2.0812	2.4974
10	1.7858	2.3216	9	2.0812	2.2893
11	1.7858	2.5000	10	2.0812	1.6650

ELASTIC MODULUS E = 0.7031×10^6 kg/cm^2

Fig. 1 Sample problem.

Table 1 Natural frequency comparison (rad/sec)

Mode no.	Exact frequency	Original Frequency ω_0	Original % error $\lvert\omega-\omega_0\rvert/\omega$	First-order perturbation $\epsilon\omega_I$	Updated value Frequency $\omega_0+\epsilon\omega_I$	Updated value % error $\lvert\omega-\omega_0-\epsilon\omega_I\rvert/\omega$
1	6.4084	7.1466	11.52	−0.8714	6.2752	2.078
2	39.275	45.015	14.62	−6.5979	38.419	2.19
3	115.30	126.60	9.80	−11.9877	114.61	0.60
4	227.90	249.06	9.52	−23.2127	225.85	0.90
5	370.01	412.54	11.49	−45.6223	366.92	0.84
6	549.89	614.86	11.82	−71.5189	543.34	1.19
7	764.99	849.05	10.99	−89.4195	759.63	0.70
8	993.94	1097.90	10.46	−111.4850	986.40	0.76
9	1221.70	1327.60	8.67	−98.3450	1229.30	0.62
10	1359.90	1490.80	9.63	−173.2830	1317.50	3.12

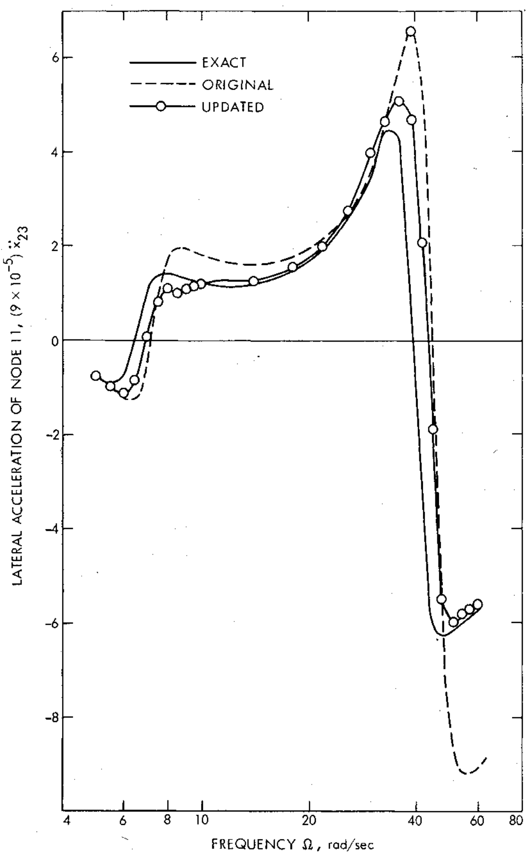


Fig. 2 In-phase response.

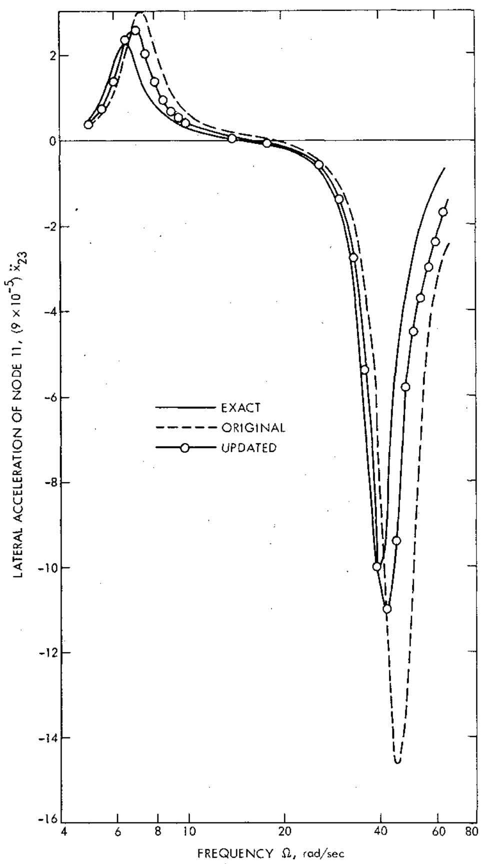


Fig. 3 Out-of-phase response.

Transient Forcing Functions

The dynamic environment of a spacecraft during launch and staging can be characterized by transient forcing functions. These forcing functions generally are characterized as high level with very short durations. In the following we will use a very short duration step function as the forcing function.

$$F(t)=0 \qquad 0\leq t<\tau \tag{39a}$$

$$F(t)=F(\tau) \qquad \tau<t<(\tau+\Delta\tau) \tag{39b}$$

$$F(t)=0 \qquad \tau+\Delta\tau<t \tag{39c}$$

From Eq. (31), the solution of a zero-order equation can be expressed as

$$u_0(\xi)=\frac{F(\tau)\cdot\Delta\tau}{\omega_0(1-p^2)^{1/2}}e^{-\rho\omega_0(\xi-\tau)}\sin[(1-\rho^2)^{1/2}\omega_0(\xi-\tau)]$$

For

$$t>\tau+\Delta\tau$$

Again, our objective is to solve u_I as a function of u_0. Therefore, let

$$\tau_I=-\frac{\omega_I}{\omega_0}\xi, \quad \frac{d\tau_I}{d\xi}=-\frac{\omega_I}{\omega_0}, \quad \frac{d^2\tau_I}{d\xi^2}=0 \tag{40}$$

and note that $dF/d\xi=0$ in this case. Upon substitution, Eq. (32) becomes

$$\frac{d^2u_I}{d\xi^2}+2\rho\omega_0\frac{du_I}{d\xi}+\omega_0^2u_I=-2\frac{\omega_I}{\omega_0}\left(\frac{d^2u_0}{d\xi^2}+2\rho\omega_0\frac{du_0}{d\xi}+\omega_0^2u_0\right) \tag{41}$$

and the solution is

$$u_I(\xi)=-2(\omega_I/\omega_0)u_0(\xi) \tag{42}$$

Table 2 Orthogonality check

Mode	1	2	3	4	5	6	7	8	9	10
1	1.00	0.023	0.019	0.014	0.006	−0.0018	0.0016	0.0023	0.0002	−0.0032
		(0.046)[a]	(−0.036)	(0.030)	(0.007)	(−0.0028)	(0.00)	(−0.002)	(0.0045)	(0.0095)
2		1.00	−0.014	0.002	0.007	0.012	0.0030	0.0032	0.00	−0.0092
			(0.034)	(0.0006)	(−0.027)	(−0.019)	−(0.0021)	(−0.010)	(0.0011)	(0.019)
3			1.00	0.017	0.004	0.0031	−0.0064	−0.0055	−0.0048	0.0013
				(−0.042)	(−0.0016)	(−0.024)	(0.0078)	(0.0076)	(0.013)	(−0.0075)
4				1.00	−0.016	−0.0046	0.0036	0.017	0.0031	−0.0044
					(−0.040)	(−0.013)	(−0.029)	(−0.030)	(0.0045)	(0.0030)
5					1.00	−0.021	0.012	−0.0076	0.010	0.011
						(0.045)	(−0.006)	(−0.013)	(−0.016)	(0.004)
						1.00	0.018	0.013	−0.017	0.020
							(−0.030)	(0.0085)	(−0.020)	(−0.018)
7							1.00	−0.023	−0.019	0.0042
								(0.038)	(−0.0075)	(0.011)
8								1.00	−0.006	0.029
									(0.031)	(0.028)
9									1.00	0.038
										(0.028)
10										1.00

[a] Numbers in parentheses are from original mode shapes.

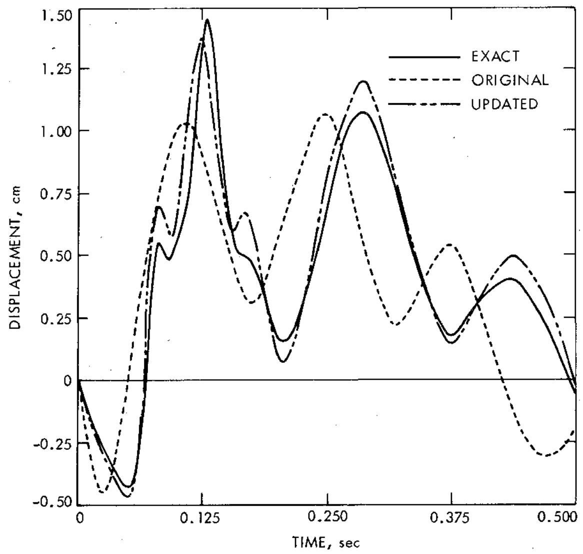


Fig. 4 Transient response of node 11.

The final solution will be

$$u(\xi) = u_0(\xi) + \epsilon u_1(\xi) = [1 - 2\epsilon(\omega_1/\omega_0)]u_0(\xi) \tag{43}$$

For transient forcing functions other than the uniform step function, the solution can be obtained by superposition, namely,

$$u_0(\xi) = \frac{1}{\omega_0(1-\rho^2)^{1/2}} \int_0^{\xi} F(\tau)e^{-\rho\omega_0(\xi-\tau)} \times \sin[(1-\rho^2)^{1/2}\omega_0(\xi-\tau)]\,d\tau \tag{44}$$

The perturbation solution expressed in Eq. (42) is still valid providing the duration of the transient is short. The updated solution may be expressed as

$$u(t) \cong \left(1 - 2\epsilon\frac{\omega_1}{\omega_0}\right)u_0\left(\frac{\omega}{\omega_0}t\right) \tag{45}$$

The first-order perturbation solutions of the generalized coordinates have been solved as functions of the original solution in the case of exponential, periodic, and transient forcing functions. These solutions can be obtained without repeating the entire solution procedure. Together with the original solution, the first-order updating of the structural response because of design modification is now available.

Results of Sample Problems

A sample problem is constructed to demonstrate the advantages and the accuracy of the method. Figure 1 shows a cantilevered beam discretized into 10 elements. Two models are constructed, Model I serves as the original model and Model II serves as the modification of Model I, and its results will be used as the exact solution. The purpose is to use the results of Model I and the known modification of mass and stiffness matrix of Model II to construct an approximate solution of Model II that will be compared with the exact solution.

Eigenvalues and Eigenvectors

Table 1 shows the eigenvalue (natural frequency) comparison of the two models toegether with the first-order perturbation updated results. The natural frequency deviation between the exact solution and the original solution is about 11% on the average. This error is reduced to about 1.3% by using the first-order perturbation modification.

The comparison of eigenvectors (normal modes) takes the form of an orthogonality check with respect to the mass matrix. Table 2 shows the orthogonality check of the updated normal modes together with that of the original model. In general, magnitudes of the off-diagonal terms are reduced – for those with higher values, the reductions are approximately 50%.

Response to Periodic Forcing Function

The response of the cantilered beam caused by a concentrated periodic force applied at node 2 has been calculated. Figures 2 and 3 show the in-phase and out-of-phase response, respectively, as a function of input frequency at node 11. Although the response profiles as shown by these two figures are similar for the original model solution and exact solution, the amplitudes near the resonance frequencies are quite different (by 50%). The first-order perturbation solution reduces this deviation to less than 10%. We already have shown that the deviations of the natural frequencies and normal modes caused by the small changes in mass and stiffness are of the order of 10%, yet the maximum response is different by 50%, which is generally unacceptable from an engineering point of view. However, in this particular case, the first-order perturbation solution is able to reduce this deviation greatly without having to reanalyze the problem.

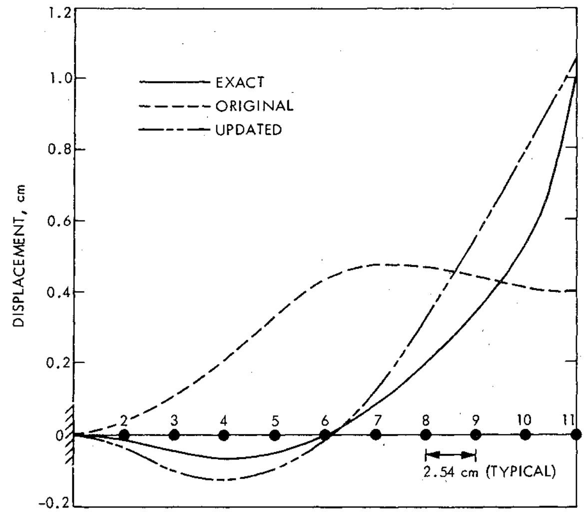


Fig. 5 Deflection shape at $t=0.3$ sec.

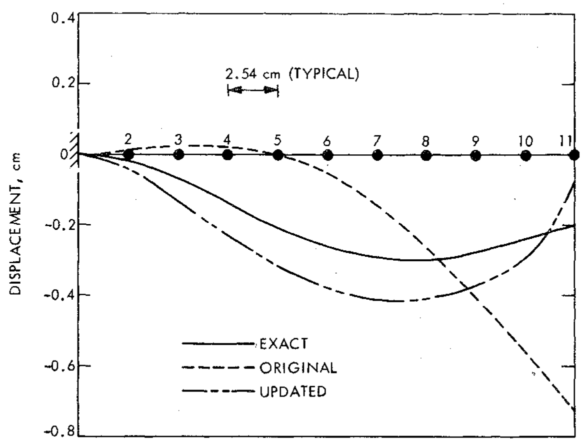


Fig. 6 Deflection shape at $t=0.6$ sec.

Response to Transient Forcing Functions

The response of the cantilevered beam caused by a concentrated transient force applied at node 2 also has been analyzed. The transient forcing function takes the form of a Dirac Delta function. Figure 4 shows the time history of node 11 under the transient forcing function for the original model solution, exact solution, and the first-order perturbation solution. Again at the maximum response, the deviation of the original solution from the exact solution is about 30%, and the first-order perturbation solution has a very small deviation from the exact solution.

The deflection shape of the beam becomes important if one wants to solve for the stresses. Figure 5 shows the deflection shapes from the original solution, the exact solution, and the first-order perturbation solution at $t=0.3$ sec. The stress distributions from the original solution and the exact solution are totally different since different modes are being excited for these two models. However, the first-order perturbation solution gives a deflection shape very close to the exact solution. Figure 6 shows similar results for the deflection shapes at $t=0.6$ sec.

Conclusion

It has been demonstrated through the sample problem that the matrix perturbation technique can be used to update the solution of structural dynamic systems because of small changes in mass and stiffness matrices. Although only the first-order perturbation solution has been considered, a substantial portion of the error can be recovered.

From the same problem, it becomes evident that small changes in mass and stiffness matrices will produce small changes in the eigenvalues and eigenvectors, but not necessarily small changes in dynamic responses. This tells the engineer to seek new solutions once the structural system is modified even slightly. The present investigation provides a method by which a new solution can be obtained without repeating the entire analysis. The methodology will be cost effective in most programs because many small changes in the structural design are made before the final design is achieved.

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